

Limit order market analysis and modelling: on an universal cause for over-diffusive prices

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Abstract

We briefly review data analysis of the Island order book, part of NASDAQ, which suggests a framework to which all limit order market models should comply. Using a simple exclusion particle model, we argue that short-time price over-diffusion in limit order markets is due to the non-equilibrium of order placement, cancellation and execution rates, which is an inherent feature of real limit order markets.

By contrast with usual data on volume and transaction price of a set of stocks or foreign exchanges [1,2,3], limit order markets provide instantaneous information about unfilled orders. In the language of Physics, one has access to an additional dimension, the price. The analysis and the modeling of such markets is therefore more complex. Economics literature has focused on the design of limit order markets (see [5,6]), asking for instance why and when limit orders may be preferred by traders [4] and introducing sophisticated models based on equilibrium prices (see for instance [7]), while physicists recently begun to analyze data and to play with dynamics stochastic toy-models where the price evolution is a consequence of stochastic order arrival [8,9,10,11,12,13,14,15,16,17].

Some definitions first. A limit order is characterized by three properties: its price p , its size m and its lifetime τ . When it is placed into the order book of a given stock, it publishes the wish to buy (or sell) m units of this stock at the predefined price p . If there is already a sell order (buy order) in the book at a lower or equal price, both orders will be automatically executed, partly or fully, depending on the size of matching opposite order(s). If this is not the case the new order will wait in the book until a compatible opposite order is placed or until its lifetime is over. This implies that at any time, there is a spatial distribution of unfilled buy and sell orders. The best prices are defined as the largest buy price and the lowest sell price. The difference between the two is called the bid-ask spread. As often emphasized, submitting a limit order is a trade-off between the advantage of obtaining a fixed price,

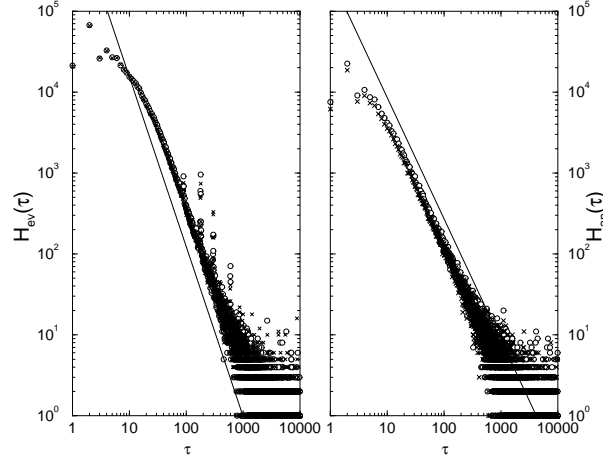


Fig. 1. Lifetime histogram for bid (x) and ask orders (circles) of all orders of all stocks and all days. The left graph plots H_{ev} , the histogram of cancelled orders (straight-line: power-law with a -2.1 exponent). The right graph plots H_{an} , the histogram of annihilated orders (straight-line: power-law with a -1.5 exponent).

and the disadvantage of not knowing precisely how long the order will have to wait.

We collected data from Island.com, a subpart of the NASDAQ. Each order has a unique ID number, which allowed us to keep track of their individual fates. Only the 15 best orders of each type were public at any given time, but we managed to find information about 80% of the orders. The major problem with this kind of data is the uncertainty in time it causes for orders that are not often listed in the 15 best ones.

We measured first order lifetime distribution, shown in Fig. 1. For both the cancelled and the executed orders, we find that its tail is well fitted by a power-law $P(\tau) \sim \tau^{-\alpha}$ with $\alpha \simeq 2.1$ for cancelled orders and $\alpha \simeq 1.5$ for annihilated orders for $\tau \leq 1000$ and then tend to broaden. This last effect is clearly due to the nature of our data: it attributes a too long lifetime to orders that are seldom seen, i.e. far from best prices. Characteristic lifetimes of 90, 120 and 180 seconds clearly appear on left panel, while the bulk of the distribution is due to active order canceling, that is, to traders watching the evolution of the market and cancelling their orders before their predefined lifetime is reached. It is tempting to relate this power-law to the dependence of the evaporation rate on the relative position, which is also a power-law [20].

We next turn to the measure of deposition (order placement), annihilation (market orders), and evaporation (order cancellation) rates, denoted δ , α and η respectively, which highly fluctuate during a trading day. Note that deposition and evaporation are proportional as a first approximation [12]. All these rates have algebraically decreasing cross-correlation, which is expected for annihilation rates, as it corresponds to that of the volume of transaction [2]. In

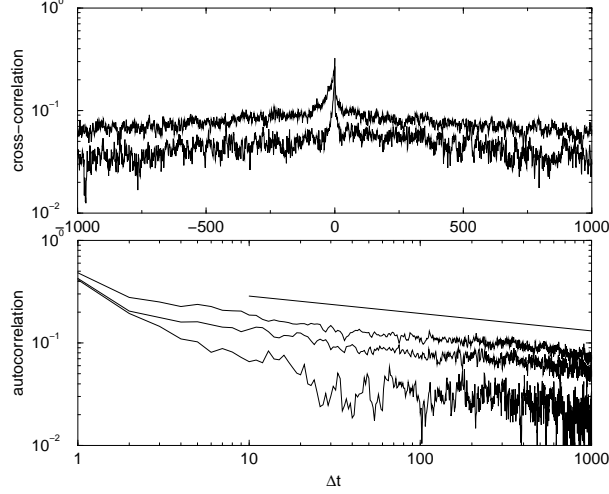


Fig. 2. Bottom graph: autocorrelation of the bid rates δ (top), η (middle) and α (bottom) measured with $\Delta t = 1$ s the dashed line has a -0.17 exponent. Upper graph: crosscorrelations $\langle \delta(t)\eta(t + \Delta t) \rangle$ (top) and η , $\langle \delta(t)\alpha(t + \Delta t) \rangle$

addition, the asymmetry in the deposition-evaporation cross-correlation function implies that evaporation triggers deposition: a trader that cancels an order is likely to wish to place it again [12]. However, we could not see trace of massive order diffusion, which is the fundamental assumption of the family of models found in Refs [8,9].

We found a linear dependence of the relative deposition width on the bid-ask spread, which can be explained by orders deposited at half of the spread, or directly at the tick adjacent to the best opposite best price.

The above observations leave us with the following framework [12] that should be the basis of any stochastic particle model of limit order markets:

- (1) bulk-deposition relative to the best price of each type of orders, with some spatial distribution. According to Refs [14,13], it is a power law;
- (2) spread-deposition with some spatial distribution;
- (3) market orders, or cross-deposition; they can be included into the previous ingredient.
- (4) cancellation of orders.

The last ingredient is crucial in these kind of model, and was introduced in [12]. Its presence ensures for instance that the number of orders does not diverge with time, and that the price evolution is diffusive for large times. Indeed, all particle models that assume fixed rates display under-diffusive prices [8,10,12,14,13], that is, $\sum_{t=1}^T \delta p_t \sim T^\beta$ with $\beta < 1/2$, where δp_t is the price increment at time t , with a crossover to diffusive prices ($\beta = 1/2$) as soon as the orders can evaporate [12]. Note also that evaporation is responsible for the crossover from $\beta = 2/3$ and $\beta = 1/2$ in the model proposed by Ref. [15].

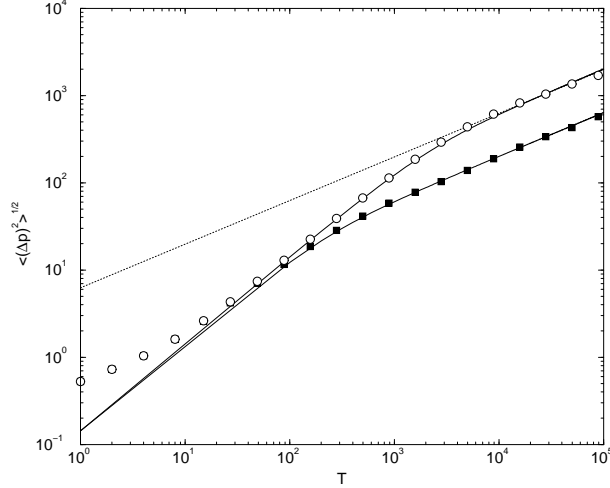


Fig. 3. Hurst plot of the price increments for probabilities of rate change $p = 0.001$ (circles) and 0.01 (squares). Deposition rates were drawn between 0 and 1 , and annihilation rates between 0 and 0.2 . No evaporation. 10^6 iterations. The dotted line represents normal random walk ($\beta = 1/2$) and continuous lines are obtained from Eq. (1).

However, the rates depend on time in real markets in such a way that the rate imbalance, that can be defined as the difference of $\delta - \alpha - \eta$ between asks and bids, is never zero. In particle models, this induces trends which lead to overdiffusive behaviours ($\beta > 1/2$) [16]. The possibly simplest model with time varying rates is an particle exclusion model where the rates are changed all at the same time [15,16]. In exclusion models of particles, only one particle can stand on a given site at a given time. Some of these models are exactly solvable [18,19]. They provide an ideal playground to test ideas on models of limit order markets.

The simplicity of these kind of models makes it possible to compute exactly some quantities, such as the Hurst exponent $\beta = 2/3$ in [15]. An analogy between price changes in the model of Ref [16] and a generalized Ising spin $1 - d$ model qualitatively explains the observed overdiffusive behavior.

Here, we repeat the argument of Ref [16] by showing numerically that time-varying rates are the key to understanding overdiffusive price behaviour in financial markets. We consider totally random and exponentially correlated rates. The only assumption is that all rates are redrawn from an uniform distribution $[0, R]$ with fixed probability p at each time step, with $R = 1$ for deposition and $R < 1$ for annihilation.¹ As shown in [16], a set of rates defines a price drift, hence a trend. Although the rates are shortly correlated, this is enough to produce overdiffusive prices, as shown by Fig. 3, and is explained by the fact that the rates stay constant over a period of time. This induces

¹ Considering the independent changes of each rate with some probability at each time step leads to the same results.

some memory in the rates themselves, consistent with the algebraical decay of rates autocorrelation (although here the decay is much faster). Neglecting bid-ask spread bounces and retaining only ballistic moves due to the drift, it is straightforward to find that price fluctuations are given by

$$\left(\sum_{t=1}^T \delta p_t\right)^2 \propto T + \frac{2(1-p)}{p^2} \left[pT - 1 + (1-p)^T\right], \quad (1)$$

which is correct for T large enough (see fig. 3). The diffusive behaviour for small T can be attributed mostly to bid-ask spread bounces and other sources of noise that were neglected in our derivation. From a practical point of view, we see no reason why the bid and ask rates should be equal at any time in real markets. This would require for instance that traders could react simultaneously to the state of the order book or to an information. Heterogeneous reaction times can be considered as a simple cause for the imbalance. Therefore, we argue that the short-term overdifusive behaviour of prices in limit order markets is quite possibly due to the unavoidable temporal imbalance of these rates. This of course does not explain long term (several months) overdifusive prices [3,2]. A better explanation may be herding behaviour [22,21], or particular economic situations.

In conclusion, we believe that Physics can contribute to the understanding of limit order markets, by asking different questions than Economics. For us, these markets are similar to non-equilibrium particle lattice systems, where the variability of deposition/annihilation bid/ask rates is the cause of short term overdifusive price evolution, and must be incorporated into current existing models, as in [16].

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